An Adaptive CUSUM-based Test for Signal Change Detection

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Abstract—Many applications, e.g., fault detection, quality of industrial process, monitoring and prediction of climatic phenomena assume the stationary hypothesis or require identification of the process change. Change detection tests satisfy the change detection necessity by identifying a drift, a different expected behavior, a deviation; their effectiveness is generally based on statistical confidence tests whose parameters are configured at design-time (generally through a trial-and-error approach). Here, we suggest an extension of the widely-used CUSUM change detection test which improves effectiveness and timeliness in detecting changes by adaptively configuring its test parameters.

I. INTRODUCTION

Change detection tests are statistical techniques that allow us for assessing the stationarity hypothesis of a data generating process, identify a possible drift, detect abrupt and sudden changes as well as smoother aging effects [1]. These tests build statistical models without a priori knowledge of the underlying physics of the process by evaluating variations of some direct or indirect parameters.

The simplest parametric technique for testing a signal change -or hypothesis- on the basis of a difference between sample means is the Student t-test [2] (which only addresses changes of the mean value). Also regression analyses [3] can be used to detect changes but they are not very sensitive to small deviations. The change detection problem can also be solved by means of a Bayesian analysis [4] when a mathematical model of the data is available.

Assumptions underlying the use of parametric tests cannot always be met in real applications. The Mann-Whitney U test for independent samples [5] and the Wilcoxon signed-rank test for related samples [6] represent non-parametric alternatives to the t-test. The U test is appropriate when we can monotonically rank two independent samples of observations (i.e., we say which one is larger). The Wilcoxon signed-rank test is a non-parametric alternative to the paired Student's t-test. The main limit of these tests is that they were originally designed for detecting single point-changes. Differently, the Mann-Kendall test [7] and the CUSUM one [8] are non-parametric tests particularly suitable in sequential analysis.

In particular, the Mann-Kendall test, which falls under the class of rank tests, evaluates the sign of all pairwise differences of observed values. The test is widely used in the analysis of climate changes and is easy to implement.

The CUmulative SUM test is based on the cumulative sums charts [9] and has been proposed to detect systematic changes over time in one or more measured variables. It provides several advantages such as a relative simplicity, a graphical interpretation of results and the ability to detect unusual patterns. It has been successfully used e.g., [8], in fault detection, onset detection in seismic signal processing, and detection of changes in mechanical systems.

Change detection tests commonly require a configuration phase to fix test parameters performed at design-time. In general, these values interact and indirectly affect the test; a trial-and-error approach is then generally used for their identification.

Here, we provide an extension of the CUSUM test which allows the test parameters to be configured. This results in an adaptive test procedure which is particularly effective when the parameters are not available a-priori or when the correct parameter configuration and the pdf of the envisaged signal are unknown.

Modifications of the CUSUM test and the parameters configuration are discussed in Section II. Section III compares the performance of Mann-Kendall, CUSUM and adaptive CUSUM in terms of accuracy, change detection readiness and computational complexity. Experiments involving the tests are finally given in Section IV on a real application.

II. LEARNING THE CUSUM TEST PARAMETERS

Both the Mann-Kendall and the CUSUM tests have parameters to be fixed at design-time; such parameters allow the tests for detecting changes in the quantity under investigation. In particular, the Mann-Kendall test requires setting a level of significance for the test (to accept a change detection at a fixed level of significance), while the CUSUM needs to fix the thresholds used to detect the possible changes in the statistical behavior. In the following we focus on the CUSUM and how to automatically estimate the optimal CUSUM thresholds.

Consider a stochastic process composed of a sequence of independent random variables \( X = \{x_1, x_2, \ldots, x_N\} \) with probability density function \( p_\theta(x) \) parameterized in the
parameters vector $\theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$. Assume that the stochastic process changes its statistical behavior at unknown time $t_0$ by maintaining its pdf model family; this is generally modeled by considering a transition from $\theta_0$ to $\theta_1$.

To measure the discrepancy between the two pdfs at time $t$ consider the log-likelihood ratio

$$s_t = \ln \frac{p_{\theta_1}(x_t)}{p_{\theta_0}(x_t)} \text{ for each } 1 \leq t \leq N$$

and evaluate the cumulative sum $S_t = \sum_{i=1}^{t} s_i$.

CUSUM identifies a change in $X$ at time $i$ when the difference $g_i$ between the value of the cumulative sum $S_i$ and its current minimum value $m_i$ at time $i$ is larger than a given threshold value $h$

$$g_i = S_i - m_i \geq h_{i\tau}, \quad m_i = \min_{1 \leq i \leq t}(S_i).$$

Very rarely parameters $\theta_0$, $\theta_1$, and $h$ are available at design-time. When this does not happen the designer has to provide an estimate based on a trial and error basis. In the following, we suggest a configuration procedure which allows the designer for automatically and adaptively identify the needed parameters. The procedure can be summarized as:

1. Define the configuration sequence;
2. Estimate the $\theta_0$ parameter;
3. Evaluate the $\theta_1$ parameter;
4. Evaluate the $h$ parameter.

Consider the first $K$ instances of $X$ (with $K < N$) as configuration sequence: $TS = \{x_i : 1 \leq i \leq K\}$.

To compute the log-likelihood ratio the knowledge of probability density function of the stochastic process is required. If it is not available, as it is mostly the case, we generate a cumulative sequence $Y = \{y_1, y_2, \ldots\}$ from $X$

$$y_1 = \frac{1}{n} \sum_{i=1}^{n} x_i; \quad y_2 = \frac{1}{n} \sum_{j=n+1}^{2n} x_j; \ldots \text{ (with } n \geq 25).$$

which is ruled by a Gaussian pdf (from the central theorem of statistics) parameterized in $\theta = \{\mu, \sigma^2\}$ (mean and variance). Such parameters can be estimated by evaluating the sample mean $\overline{Y}$ and the variance $S^2$ of the instances belonging to the configuration sequence.

We then apply the CUSUM test to $\overline{Y}$ by measuring the “distance” $s_t$ time by time. By considering both mean and variance as parameters, we can detect four possible changes in the statistical behavior of process $Y = \{y_1, y_2, \ldots\}$ as shown in Table 1.

**TABLE 1. POSSIBLE CHANGES**

<table>
<thead>
<tr>
<th>Changes</th>
<th>$\Delta\mu = 0$</th>
<th>$\Delta\mu \neq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta\sigma = 0$</td>
<td>No change</td>
<td>Additive change</td>
</tr>
<tr>
<td>$\Delta\sigma \neq 0$</td>
<td>Spectral (nonadditive) change</td>
<td>Spectral and additive change</td>
</tr>
</tbody>
</table>

Starting from $\theta_0 = \{\overline{Y}, S^2\}$, if we want to detect an additive change we have to compute an alternative hypothesis $\overline{Y}'$ hence assuming a $\theta_1 = \{\overline{Y}', S^2\}$ configuration. Conversely, if we want to detect a spectral change, we have to compute the hypothesis $S^2'$ for the variance value in $p_{\theta_1}(y_t)$. In case of spectral and additive change detection we must provide both hypothesis: $\theta_1 = \{\overline{Y}', S^2'\}$ for $p_{\theta_1}(y_t)$. In the remaining of the section we show how to compute the most general case $\overline{Y}$ and $S^2'$.

Once estimated the sample mean and variance, it is possible to compute a $(1-\alpha)\%$ confidence interval $[\overline{Y}_{\text{min}}, \overline{Y}_{\text{max}}]$ for $\overline{Y}$, e.g., with $\alpha = 0.01$.

We then compute the distance

$$d_T = |\overline{Y} - \overline{Y}_{\text{min}}| = |\overline{Y} - \overline{Y}_{\text{max}}| \text{ between } \overline{Y} \text{ and the extremes of the confidence interval } [\overline{Y}_{\text{min}}, \overline{Y}_{\text{max}}].$$

Parameter $\overline{Y}'$ becomes $\overline{Y}' = \overline{Y} \pm M * d_T$ with $M$ being the sensitivity of the test. In fact, by increasing its value, $\overline{Y}'$ diverges from $\overline{Y}$ and the test is less sensitive to small fluctuations of the random variable. Conversely, by decreasing $M$, the test is able to recognize very quickly the beginning of a change. The choice of $M$ hence influences the detection ability (it is easy to prove that by fixing $M$ the amplitude of the minimum detectable drift is $\frac{M * d_T}{2}$).

The presence of the symbol $\pm$ suggests that in addition to the identification of a change we are also able to detect an increasing change (in case of $+$) or a decreasing one (in case of $-$). If we want to detect both changes we have simply to set up two tests with two distinct configuration phases, one for the increasing and one for the decreasing change.

From theory we know that

$$[S^2_{\text{min}}, S^2_{\text{max}}] = \left[\frac{nS^2}{\chi^2_{\alpha/2}}, \frac{nS^2}{\chi^2_{1-\alpha/2}}\right].$$
is a $100(1 - \alpha)\%$ confidence interval for the variance. We can then compute $d_{g_2}$ and $S_2 = S_1^2 \pm M * d_{g_2}$.

Once identified the parameters in $\theta_0$ and in $\theta_1$, we proceed to compute $S_t$, $m_t$, and $g_t$ for $1 \leq t \leq K$. Finally, the maximum value of $g_t$ in the configuration sequence is used as threshold value $h$ for the test $h = \max_{1 \leq i \leq K} |S_i|$. CUSUM test can now be applied.

The test can be easily made adaptive hence following the process deviations by foreseen an auto-configurations of its parameters $\theta_0$, $\theta_1$ and $h$ during its operational life. These auto-configurations could happen at specific time-instances or at the verification of specific conditions (e.g. a drift is detected).

III. TEST COMPARISON

To compare the performance of the Mann-Kendall test, the original CUSUM test and the adaptive CUSUM test, we generated a simulated change detection experiment and focused on the detection of additive changes. A Gaussian process was defined that changes its mean value at a specific time instance. (The mean value of the Gaussian process is $\mu_0 = 100$ for the first 500 samples and becomes $\mu_i = 105$ for other 500). The standard deviation of the process was $\sigma = 3$.

We evaluated the change detection tests on the obtained sequence resulting in 1000 binary values 0 (no change detected) or 1 (change detected). In the optimal detection case we shall have 0s for the first 500 samples and then 1s.

The Mann-Kendall test was set up by only fixing the level of significance at 95%. The original CUSUM test was configured by using the known values $CUSUM_{\mu_0} = \mu_0$ and $CUSUM_{\mu_1} = \mu_1$ (which represents the best condition with a priori information). Our adaptive CUSUM test was configured using the first 250 samples. The parameter $M$ was fixed at 10.

We evaluated 4 figures of merit to compare the performance of tests:
- False Positive (FP) counts the times a test detects a change in the sequence when there is not,
- False Negative (FN) counts the times a test does not detect a change when there it is,
- Recognition Capability Speed (RCS) measures the speed to detect the presence of the change by counting the delay time in detection;
- Computational Time (CT) evaluates the computational complexity needed to perform the change detection.

The experiment was repeated 1000 times and results averaged as presented in TABLE II.

It is worth noting that the adaptive CUSUM test guarantees results comparable with the original CUSUM but does not require any a priori information (moreover, here CUSUM operates in the ideal case by possessing the true mean and variance values). The Mann-Kendall test tended to provide many false negative and shows to be generally slow in detecting a change. Moreover, the CUSUM tests are characterized by faster execution.

<table>
<thead>
<tr>
<th>Test</th>
<th>Figures of merit</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>FP (%)</td>
</tr>
<tr>
<td>MK</td>
<td>0.88</td>
</tr>
<tr>
<td>CUSUM</td>
<td>0.59</td>
</tr>
<tr>
<td>Adaptive CUSUM</td>
<td>0.82</td>
</tr>
</tbody>
</table>

We then evaluated the relationship between the parameter $CUSUM_{\mu_1}$ and its change detection ability (simpistic sensitivity analysis) hence resembling the case of a non-precise hypothesis definition for $\mu_1$; results are given in Table III. As expected, by increasing $CUSUM_{\mu_1}$ from $\mu_0 = 100$, FP and RCS increase. Conversely, when $CUSUM_{\mu_1}$ decreases, FP increases.

<table>
<thead>
<tr>
<th>$CUSUM_{\mu_1}$</th>
<th>FP (%)</th>
<th>FN (%)</th>
<th>RCS (samples)</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>1.0984</td>
<td>1.4194</td>
<td>8.069</td>
</tr>
<tr>
<td>103</td>
<td>0.8732</td>
<td>0.6452</td>
<td>4.155</td>
</tr>
<tr>
<td>105</td>
<td>0.5932</td>
<td>0.6078</td>
<td>3.791</td>
</tr>
<tr>
<td>107</td>
<td>0.4234</td>
<td>0.8472</td>
<td>4.329</td>
</tr>
<tr>
<td>109</td>
<td>0.2534</td>
<td>2.6152</td>
<td>5.426</td>
</tr>
</tbody>
</table>

IV. EXPERIMENTAL RESULTS

In X-ray applications the drift of the X-ray source over time can be suitably measured by photodiodes. Each photodiode has its specific fluctuations, mainly due to thermal noise.

Is it possible to understand whether there is a non-stationarity in the process, e.g., due to the X-ray source change or not?

We considered a set of 320 photodiodes to study their group behavior and apply the adaptive CUSUM test to verify if the group was exhibiting a change induced by some non-stationarity phenomenon. A campaign of 13000 samples was generated. An initial preprocessing of the data was performed to compute the sample mean of consecutive subsequences composed by 25 measurements. As explained in Section II, this initial step aims at creating a new sequence of measurements that belong to a Gaussian process (starting from a process with unknown probability density function).

We thus obtained 320 sequences of diode measurements of 520 samples length. The first 40 measurements compose the
configuration sequence and the parameter $M$ was experimentally fixed at 4.

We applied the adaptive CUSUM test to all the sequences with the aim at discovering a common behavior of the diodes. A group of 16 diodes was randomly chosen to become the reference diodes. The analysis of these diodes is expected to provide an estimate of the statistical behavior of the other non-reference diodes. After the initial configuration sequence we computed for each time instant the percentage of reference diodes $P_{ref}^{t} = \{p_{ref}^{t_{0}}, p_{ref}^{t_{1}}, p_{ref}^{t_{2}}, \ldots\}$ presenting a change according to the adaptive CUSUM. $P_{ref}^{t} = RT_{ref}^{t} / R_{ref}^{t}$ represents the number of reference diodes with change at instant $t$ w.r.t. the total number of diodes. The figure of merit computes the percentage of reference diodes presenting a change to foresee the behavior of the other non-reference diodes. We then computed for each time instant the percentage of diodes $P_{non-ref}^{t} = \{p_{non-ref}^{t_{0}}, p_{non-ref}^{t_{1}}, p_{non-ref}^{t_{2}}, \ldots\}$ presenting a change among the non-reference diodes, where $P_{non-ref}^{t} = RT_{non-ref}^{t} / R_{non-ref}^{t}$ represents the number of non-reference diodes with change at instant $t$ w.r.t. the total number of non-reference diodes. Examples of $P_{ref}$ and $P_{non-ref}$ are presented in figure 1.

To evaluate the estimate ability of reference diodes we computed $E = \{e_{t_{0}}, e_{t_{1}}, e_{t_{2}}, \ldots\}$, where $e_{i} = |P_{ref}^{t_{i}} - P_{non-ref}^{t_{i}}|$; results are given in figure 2.

The very small value of $E$ means that the difference between the percentage of reference diodes exhibiting a change and the percentage of non-reference diodes exhibiting a change is very little. Thus, these results show that the reference diodes can well infer the behavior of other diodes. Thus by analyzing only the reference diodes we can predict the behavior of the others when receiving the same radiation.

V. CONCLUSIONS

In this paper we presented an extension of a well-known CUSUM change detection test to overcome the need to configure at design-time the parameters. The automatic configuration of the parameters is proposed in Section II. A change detection experiment, presented in Section III, was set up and its results showed the advantages of the use of the adaptive approach in case of not precise parameter configuration. This approach is very appealing both when the correct parameters are not available a-priori and when, as presented in Section IV, neither the correct parameter configuration nor the pdf of the signal is a-priori known.

REFERENCES